

# Fermion localization on degenerate and critical branes

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## Abstract

In this work we analyze the localization of fermions on degenerate and critical Bloch branes. This is done directly on physical coordinates, in contrast to some works that has been using conformal coordinates. We find the range of coupling constants of the interaction of fermions with the scalar fields that allow us to have normalizable fermion zero-mode localized on the brane on both, critical and degenerate Bloch branes. In the case of critical branes our results agree with those found in [Class. Quantum Grav. **27** (2010) 185001]. The results on fermion localization on degenerate Bloch branes are new. We also propose a coupling of fermions to the scalar fields which leads to localization of massless fermion on both sides of a double-brane.

## 1 Introduction

The idea that the Universe we live in can be realized by a static 3-domain wall (3-brane) immersed in a (4,1)-dimensional world has opened a pathway for the localization of matter [1] and gauge bosons [2] in worlds with large extra dimensions without resorting to mechanisms of compactification [3]. Large extra dimensions has also provided mechanisms to solve the hierarchy of interactions problem [4]-[7] as well as the cosmological constant problem [8]. In [9] it was shown that the effective gravitational potential between two particles recovers the Newtonian behavior, since one has localization of gravitons on a thin brane in five-dimensional space-time with an warped geometry and the cosmological constant is related to the brane tension. Later the localization of matter (spin-zero, spin-1/2 and spin-3/2) in the Randall-Sundrum framework was shown to be possible, under certain conditions over the brane tension [10]. It is important to emphasize that the localization of fermions on thin branes is, in fact, provided by an *ad hoc* soliton and the mechanism for localization follows

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straightforwardly *à la* Jackiw and Rebbi [11]. As a matter of fact, it is such a soliton that provides the domain-wall and the fermion localization in the scenario proposed by Rubakov and Shaposhnikov [1].

By introducing a non-linear model with a set of scalar fields, in 5-dimensional space-time with warped geometry, one has a set of coupled non-linear differential equations whose minimum energy solutions are thick branes and self-consistent warp factors. The thick branes also separate the space in two patches characterized by a peculiar warp factor whose asymptotic behavior is an anti-de Sitter ( $\text{AdS}_5$ ) space [12]-[19], as in the Randall-Sundrum framework (thin brane), but without singularities. The stability of such solutions is very difficult to be proven due to the intricate differential equations one has to deal with [17]. Notwithstanding, one still has localization of gravitons on thick branes, as has been shown in [18]. In fact, this program has been developed as a generalization of the domain wall universe, by taking into account the stabilization of gravity fluctuations via domain-walls in supergravity theories [21].

Localization of matter on thick branes has been illustrated by using several different non-linear models for scalar fields coupled with gravity [22]-[24]. One of those models is the Bloch branes model [25], which comprises two interacting scalar fields whose classical solutions are the Bogomolnyi-Prasad-Sommerfield (BPS) and the warp factor can also be obtained as solution of a first-order differential equation. Moreover, the model exhibits a richer structure due to the variety of kinks (solitons) it comprises [26]-[28], leading to what has been called degenerate and critical Bloch branes [29], with a self-consistent warp factor and localization of gravitons. A natural track that has been followed is the localization of matter in such a variety of branes.

In [30] the localization of massless fermions has been studied together with an analysis of resonant fermion modes with a specific coupling of fermions to the classical BPS configurations of Bloch branes and in [31] it is studied the localization of massless fermions on critical Bloch branes. It has been explicitly pointed out [29] that critical Bloch branes arise as a critical limit of degenerate Bloch branes due to a running, but limited, constant of the integration of the orbit equation relating the classical configurations for the scalar fields. Thus the localization of massless fermions on degenerate and critical branes is an important question to be analyzed. As a matter of fact, a Bloch brane may be seen as a thick brane that evolves to a thicker one, namely, a degenerate Bloch brane which, in its turn, splits into two branes whose separation becomes large as the degeneracy parameter approaches the critical value, at which the critical brane is triggered. Such a picture resembles the description of first order phase transitions which was also used in the context of brane worlds to describe brane splitting [32]. In fact, as a counterpoint, in the model we use here, the free energy itself does not depend on the temperature; instead, this dependence is implicit in the degeneracy parameter.

The objective of the present work is to analyze the fate of massless fermions trapped on a split brane. In section II we give a brief review of the model to be used, together with the consistent branes and warp factors solutions. In section III we deal with the localization of massless fermions on the critical and degenerate branes and the IV section is devoted to the conclusions and further remarks.

## 2 A brief review of a model and its degenerate and critical Bloch branes

The action in five-dimensional gravity coupled to two interacting real scalar fields, can be represented by

$$S = \int d^4x dr \sqrt{|g|} \left[ -\frac{1}{4}R + \frac{1}{2}(\partial_\mu \phi \partial^\mu \phi + \partial_\mu \chi \partial^\mu \chi) - V(\phi, \chi) \right], \quad (1)$$

where  $g \equiv \det(g_{ab})$  and

$$ds^2 = g_{ab} dx^a dx^b = e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu - dr^2, \quad (a, b = 0, \dots, 4), \quad (2)$$

where  $r$  is the extra dimension,  $\eta_{\mu\nu}$  the usual Minkowski metric in the four space-time dimensions and  $e^{2A(r)}$  is the so-called warp factor.

If the potential  $V(\phi, \chi)$  can be written in terms of a superpotential as

$$V(\phi, \chi) = \frac{1}{2} \left[ \left( \frac{\partial W(\phi, \chi)}{\partial \phi} \right)^2 + \left( \frac{\partial W(\phi, \chi)}{\partial \chi} \right)^2 \right] - \frac{4}{3} W(\phi, \chi)^2, \quad (3)$$

By substituting the superpotential, and using the orbit equation, we obtain

$$A(\chi) = \alpha_0 + \left( \frac{2\lambda a^2}{9\mu} \right) \ln(\chi) - \frac{1}{9} \left( \frac{\lambda - 3\mu}{\lambda - 2\mu} \right) \chi^2 - \left( \frac{c_0}{9} \right) \chi^{\lambda/\mu}, \quad (\lambda \neq 2\mu), \quad (4)$$

$$A(\chi) = \alpha_1 + \left( \frac{2\lambda a^2}{9\mu} \right) \ln(\chi) - \frac{(3\mu + \lambda c_1)}{18\mu} \chi^2 - \frac{1}{6\mu} \chi^2 \left( \ln(\chi) - \frac{1}{2} \right) \quad (\lambda = 2\mu), \quad (5)$$

where  $\alpha_0$  and  $\alpha_1$  are arbitrary integration constants, which are chosen to be  $A(r=0) = 0$ .

It has also been found in [28, 29] that the classical solutions for  $c_0 < -2a$  and  $\lambda = \mu$  are given by

$$\chi_{DBW}^{(1)}(r) = \frac{2a^2}{\left( \sqrt{c_0^2 - 4a^2} \right) \cosh(2\mu ar) - c_0}, \quad (6)$$

$$\phi_{DBW}^{(1)}(r) = \frac{a \left( \sqrt{c_0^2 - 4a^2} \right) \sinh(2\mu ar)}{\left( \sqrt{c_0^2 - 4a^2} \right) \cosh(2\mu ar) - c_0}. \quad (7)$$

The corresponding warp factor is expressed as

$$e^{2A(r)} = N \left[ \frac{2a^2}{\left( \sqrt{c_0^2 - 4a^2} \right) \cosh(2\mu ar) - c_0} \right]^{4a^2/9} \times \exp \left\{ \frac{2a^2 \left[ c_0^2 \pm 4a^2 - c_0 \left( \sqrt{c_0^2 - 4a^2} \right) \cosh(2a\mu r) \right]}{9 \left[ \left( \sqrt{c_0^2 - 4a^2} \right) \cosh(2a\mu r) - c_0 \right]^2} \right\}, \quad (8)$$

where  $N$  is chosen such that  $e^{2A(0)} = 1$ , for plotting convenience.

On the other hand, for  $\lambda = 4\mu$  and  $c_0 < 1/(16a^2)$  the solutions can be written as

$$\chi_{DBW}^{(2)}(r) = -\frac{2a}{\sqrt{(\sqrt{1-16c_0a^2}) \cosh(4\mu ar) + 1}}, \quad (9)$$

$$\phi_{DBW}^{(2)}(r) = \frac{a(\sqrt{1-16c_0a^2}) \sinh(4\mu ar)}{(\sqrt{1-16c_0a^2}) \cosh(4\mu ar) + 1}. \quad (10)$$

with the warp factor

$$e^{2A(r)} = N \left[ -\frac{2a}{\sqrt{(\sqrt{1-16c_0a^2}) \cosh(4\mu ar) + 1}} \right]^{16a^2/9} \times \\ \times \exp \left\{ -\frac{4a^2}{9} \left[ \frac{1 + 8a^2c_0 + (\sqrt{1-16a^2c_0}) \cosh[4\mu ar]}{(1 + (\sqrt{1-16a^2c_0}) \cosh[4\mu ar])^2} \right] \right\}. \quad (11)$$

The set of solutions above, was baptized by Dutra and Hott [29] as degenerate Bloch walls (DBW).

Furthermore, an interesting class of analytical solutions, named as critical Bloch walls (CBW), was shown to exist when the constant of integration equals a critical value. For  $\lambda = \mu$  and  $c_0 = -2a$ , one has the set of solutions for the scalar fields

$$\chi_{CBW}^{(1)}(r) = \frac{a}{2} [1 \pm \tanh(\mu ar)], \quad (12)$$

$$\phi_{CBW}^{(1)}(r) = -\frac{a}{2} [\tanh[\mu ar] \mp 1], \quad (13)$$

which leads to the following warp factor

$$e^{2A(r)} = N \left\{ \frac{a}{2} [1 \pm \tanh(\mu ar)] \right\}^{2a^2/9} \times \\ \exp\left(\frac{a^2}{9} [1 - \tanh^2(\mu ar)]\right). \quad (14)$$

For  $\lambda = 4\mu$  and  $c_0 = 1/(16a^2)$  the solutions for the fields are given by

$$\chi_{CBW}^{(2)}(r) = \sqrt{2} a \frac{\cosh(\mu ar) \pm \sinh(\mu ar)}{\sqrt{\cosh(2\mu ar)}}, \quad (15)$$

$$\phi_{CBW}^{(2)}(r) = \frac{a}{2} [\pm 1 - \tanh(2a\mu r)], \quad (16)$$

and the warp factor is

$$e^{2A(r)} = N \left[ \frac{2 a^2 e^{\pm 2\mu ar}}{\cosh(2\mu ar)} \right]^{8a^2/9} \times \\ \times \exp \left\{ \frac{2a^2}{9} \frac{e^{\pm 2\mu ar}}{\cosh(2\mu ar)} \left[ 1 + \frac{e^{\pm 2\mu ar}}{4 \cosh(2\mu ar)} \right] \right\}. \quad (17)$$

In Fig.1 are shown profiles of the warp factor in the case of DBW with  $\lambda = 4\mu$  for some values of the constant of integration  $c_0$  and in Fig. 2 the behavior of the warp fact in the case CBW and  $\lambda = 4\mu$  is shown for some values of the parameter  $a$ .

We would like to warn the reader that the expressions for the warp factor and the Figures 2 and 3 presented in [29] do not agree with those presented here. The correct expressions and Figures are those presented here. Despite this mistake, the conclusions of the work [29] are not wrong, except for the fact that we had attributed the appearance of two peaks in the warp factor, presented in figure 2 of that work, as a sign of the formation of two domain walls. As a matter of fact, two-kink solutions can be seen from the expressions (7) and (10) for values of  $c_0$  close to the critical value. In Fig 3. we illustrate the two-kink solution with  $\lambda = 4\mu$  for two values of the constant of integration  $c_0$ . In brane cosmology scenario we are interested in, the formation of two branes leads to an almost flat space-time region between them, that is, the warp factor does vary appreciably between the two branes, as can be depicted from Figure 1. The critical brane could be seen as two branes, but infinitely separated from each other, such that one has a complete wetting.

The two-kink solutions of this model were recently investigated to discuss the phenomenon of brane splitting by means of an effective model with only one scalar field [33]. In fact, that effective model was built based on the very same model explored here . The constant of integration  $c_0$  plays the role of a coupling constant in the effective potential obtained in [33], such that one could think of the effective potential as a free energy that describes a first-order phase transition characterized by the emergence of a growing wet (disordered) phase in between two ordered phases [32]. The branes are the domain walls separating the disordered domain from the ordered ones, as can be seen from the behavior of the energy density of the matter fields.

### 3 Localization of fermions

In this section, we study the localization of massive fermions on the degenerate and critical Bloch branes [29]. For this, we consider a Dirac spinor field coupled with the scalar fields by a general Yukawa coupling. Thus, the action we are going to work is given by

$$S_{1/2} = \int d^5x \sqrt{-g} \left( \bar{\Psi} i \Gamma^a D_a \Psi - \eta \bar{\Psi} F(\phi, \chi) \Psi \right), \quad (18)$$

consequently the equation of motion is

$$[i \Gamma^a D_a - \eta F(\phi, \chi)] \Psi = 0, \quad (19)$$

where  $F(\phi, \chi)$  is a functional of the classical configurations which are solutions of the equations (??). The equation of motion for the fermion can be rewritten as

$$[i\Gamma^\mu D_\mu + i\Gamma^4 D_4 - \eta F(\phi, \chi)]\Psi = 0. \quad (20)$$

The relations between the warped-space gamma matrices ( $\{\Gamma^a, \Gamma^b\} = 2g^{ab}$ ), with  $g^{ab}$  defined in (2), and the Minkowskian ones ( $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ ) can be realized as follows

$$\Gamma^\mu = e^{-A(r)}\gamma^\mu \text{ and } \Gamma^4 = -i\gamma^5. \quad (21)$$

Moreover, we have the following expression for the covariant derivative

$$D_a = (\partial_a + \omega_a) = \partial_a + \frac{1}{4}\omega_a^{\bar{a}\bar{b}}\Gamma_{\bar{a}}\Gamma_{\bar{b}}, \quad (22)$$

where  $\bar{a}$  and  $\bar{b}$ , denote the local Lorentz indices. Thus the spin connection  $\omega_a^{\bar{a}\bar{b}}$  is given by

$$\omega_a^{\bar{a}\bar{b}} = \frac{1}{2}E^{\bar{b}\bar{a}}(\partial_a E_{\bar{b}}^{\bar{a}} - \partial_{\bar{b}} E_a^{\bar{a}}) - \frac{1}{2}E^{\bar{b}\bar{c}}(\partial_a E_{\bar{c}}^{\bar{a}} - \partial_{\bar{c}} E_a^{\bar{a}}) - \frac{1}{2}E^{\bar{c}\bar{a}}E^{\bar{b}\bar{p}}(\partial_c E_{\bar{p}}^{\bar{a}} - \partial_p E_c^{\bar{a}})E_a^{\bar{q}}. \quad (23)$$

In the above definition  $E_a^{\bar{a}}$  is the vielbein, and the non-vanishing components of  $\omega_a$  are

$$\omega_\mu = \frac{1}{2}e^A(\partial_r A)\gamma_\mu\gamma_5. \quad (24)$$

Then the equation of motion for the fermion is given by

$$\{i\gamma^\mu\partial_\mu + e^{A(r)}\gamma_5[\partial_r + 2\partial_r A(r)] - \eta e^{A(r)}F(\phi, \chi)\}\Psi(x, r) = 0. \quad (25)$$

From now on we make use the general chiral decomposition

$$\Psi(x, r) = \sum_n \psi_{Ln}(x)\alpha_{Ln}(r) + \sum_n \psi_{Rn}(x)\alpha_{Rn}(r), \quad (26)$$

with  $\gamma^5\psi_{Ln}(x) = -\psi_{Ln}(x)$  and  $\gamma^5\psi_{Rn}(x) = \psi_{Rn}(x)$ . Furthermore, we assume that  $\psi_{Ln}(x)$  and  $\psi_{Rn}(x)$  satisfy the 4-dimensional massive Dirac equations

$$i\gamma^\mu\partial_\mu\psi_{Ln}(x) = m_n\psi_{Ln}(x), \quad (27)$$

$$i\gamma^\mu\partial_\mu\psi_{Rn}(x) = m_n\psi_{Rn}(x). \quad (28)$$

Thus, applying the chiral decomposition (26) in equation (25) and using equations (27) and (28), we arrive at the equations below

$$\left[\frac{d}{dr} + 2\frac{dA(r)}{dr} - \eta F(\phi, \chi)\right]\alpha_{Rn}(r) = -m_n e^{-A(r)}\alpha_{Ln}(r), \quad (29)$$

$$\left[\frac{d}{dr} + 2\frac{dA(r)}{dr} + \eta F(\phi, \chi)\right]\alpha_{Ln}(r) = m_n e^{-A(r)}\alpha_{Rn}(r). \quad (30)$$

for the  $r$ -dependent parts of the spinor  $\Psi(x, r)$ . In order to find reliable results concerning the fermion localization in the brane we make use of the following ortonormalization relations for the  $r$ -dependent parts of the spinor:

$$\int_{-\infty}^{\infty} e^{3A(r)} \alpha_{Ln}(r) \alpha_{Lm}(r) dr = \int_{-\infty}^{\infty} e^{3A(r)} \alpha_{Rn}(r) \alpha_{Rm}(r) dr = \delta_{nm}, \quad (31)$$

$$\int_{-\infty}^{\infty} e^{3A(r)} \alpha_{Ln}(r) \alpha_{Rm}(r) dr = 0. \quad (32)$$

By redefining the  $r$ -dependent parts of the spinor as

$$\alpha_{Rn}(r) = e^{-2A(r)} L_{Rn}(r), \quad \alpha_{Ln}(r) = e^{-2A(r)} L_{Ln}(r), \quad (33)$$

we are able to get rid of the second-term in the left-hand sides of the equations (29) and (30). Thus we obtain

$$\left[ \frac{d}{dr} - \eta F(\phi, \chi) \right] L_{Rn}(r) = -m_n e^{-A(r)} L_{Ln}(r), \quad (34)$$

$$\left[ \frac{d}{dr} + \eta F(\phi, \chi) \right] L_{Ln}(r) = m_n e^{-A(r)} L_{Rn}(r). \quad (35)$$

We notice that the above equations are equivalent to the equations for the components of a spinor describing a massless fermion in 1+1 dimensions subject to a mixing of scalar and vector potentials. The time-independent equation for a fermion under such potentials can be written as  $H\psi(r) = E\psi(r)$ , with the Dirac Hamiltonian given (in natural units) by  $H = \sigma_2 p + V_s(r) \sigma_1 + V_v(r)$ , where  $p = -id/dr$  is the momentum operator,  $\sigma_1$  and  $\sigma_2$  are the two non-diagonal Pauli matrices,  $V_s(r) = -\eta F(\phi, \chi)$  (note that  $F(\phi, \chi)$  is a function of  $r$ ) is the scalar potential and  $V_v(r) = m_n e^{-A(r)}$  is the vector potential. In this analogy, one can say that  $L_{Ln}(r)$  and  $L_{Rn}(r)$  play the role of the upper and lower components for the fermion zero-mode in 1+1 dimensions. We have mentioned that the fermion in 1+1 dimensions is massless, but in fact, the scalar potential can be thought as a position-dependent mass. As one knows, many examples of such systems were already solved in the literature [34], particularly when the scalar potential is proportional to the vector potential, namely  $V_s(r) = \delta V_v(r)$ , that allow for fermion bound states. The vector potential can be attractive for fermions whilst it is repulsive for antifermions and *vice-versa*, that is, one can have pair-production. On the other hand, if  $\delta \geq 1$  and the vector potential is attractive for anti-fermions, the mixing of such potentials supports bound states. This could be explained due to an increasing of the threshold for the pair-production provided by  $V_s(r)$ , since it contributes as a variable mass for the fermion and the energy provided by the electric field does not reach two times the effective mass of the fermion. In the brane world scenario we are considering here, massive ( $m_n \neq 0$ ) as well as massless ( $m_n = 0$ ) fermions might be localized inside the brane, depending on the shape and strength of the coupling with the scalar field  $\eta F(\phi, \chi)$ , although, the issue of localization of massive modes in the brane is a very difficult as discussed below.

Concerning the localization of massless fermions in the brane, one finds that the  $r$ -dependent coefficients appearing in the chiral decomposition (30) are written as

$$\begin{aligned}
\alpha_{R_0}(r) &= N_{R_0} \exp[-2A(r) + \eta \int^r F(r') dr'] \\
\alpha_{L_0}(r) &= N_{L_0} \exp[-2A(r) - \eta \int^r F(r') dr'] ,
\end{aligned} \tag{36}$$

where we have defined  $F(r) = F(\phi(r), \chi(r))$ . In general, one resorts to an analytical function  $F(\phi(r), \chi(r))$  in order to have localized massless fermions in the brane and those localized states have invariably a well defined chirality, which depends on the behavior of  $\int^{r \rightarrow \pm\infty} F(r') dr$  and on the sign of  $\eta$ , since the normalization condition for the  $r$ -dependent parts of the massless fermion are given by

$$|N_{R_0}|^2 \int_{-\infty}^{\infty} e^{-A(r) + 2\eta \int^r F(r') dr'} dr = |N_{L_0}|^2 \int_{-\infty}^{\infty} e^{-A(r) - 2\eta \int^r F(r') dr'} dr = 1. \tag{37}$$

An interesting approach of this, can be found in the work by Slatyer and Volkas [24]. In general, if  $\alpha_{R_0}(r)$  is normalizable,  $\alpha_{L_0}(r)$  is not and *vice-versa*, then the non-normalizable contribution equals to zero. This could explain why we observe neutrinos with only one chirality in our universe. The fact that only one of the chiralities is normalizable can be seen from the effective Schrödinger equation that can be derived for  $L_{R0}(r)$  and  $L_{L0}(r)$  which are zero-modes of effective potentials that are supersymmetric partners of each other, and the normalization of both components would imply into the supersymmetry breaking. One can also check that the factor  $e^{-A(r)}$  in the integrands above is also important for the normalization of the spinor, although it is not decisive for the resolution of the chirality, since it amounts to the same weight in the normalization for both chiralities.

For  $m_n \neq 0$  we observe that the left and right components can be decoupled. It can be shown that  $L_{Ln}(r)$  obeys the following second-order differential equation

$$L''_{Ln} + A' L'_{Ln} + [\eta A' F + \eta F' - \eta^2 F^2 + m_n^2 e^{-2A}] L_{Ln} = 0, \tag{38}$$

where the prime denotes derivative with respect to  $r$ . With the redefinition

$$L_{Ln}(r) = e^{-A(r)/2} f_{Ln}(r). \tag{39}$$

one finds that  $f_{Ln}(r)$  obeys a time-independent Schrödinger equation

$$-f''_{Ln} + U_{eff}^L f_{Ln}(r) = 0, \tag{40}$$

with eigenvalue equals zero and the effective potential

$$U_{eff}^L(r) = \eta^2 F^2 - \eta F' - \eta A' F + (1/4) A'^2 + (1/2) A'' - m_n^2 e^{-2A(r)}. \tag{41}$$

For the right-component one finds

$$-f''_{Rn} + U_{eff}^R f_{Rn}(r) = 0, \tag{42}$$

with

$$U_{eff}^R(r) = \eta^2 F^2 + \eta F' + \eta A' F + (1/4) A'^2 + (1/2) A'' - m_n^2 e^{-2A(r)}. \tag{43}$$



We have reduced the problem of localization of massive fermions on a brane into a Sturm-Liouville problem, which agrees with the results found in [22] and [24], without resorting to the transformation of variable as done in others papers.

One can note that there is a symmetry relating the effective potentials, namely  $U_{eff}^R = U_{eff}^L|_{\eta \rightarrow -\eta}$  and that the eigenstates  $f_{Ln}(r)$  and  $f_{Rn}(r)$  are subject to distinct effective potentials. In order to have localization of a specific massive mode  $m_n$ , each one of the chiralities must be a bound state of its corresponding potential with energy equal to zero. Due to the above symmetry and the dependence on the mass in the expressions (41) and (43) one has a specific potential for each massive mode and specific chirality. In general, the equations (40) and (42) are very difficult to be analytically solved, but there are two particular and unphysical cases, namely  $\eta F = m_n e^{-A(r)}$  and  $\eta F = -m_n e^{-A(r)}$ , for which one can find exact expressions for  $\alpha_{Rn}(r)$  and  $\alpha_{Ln}(r)$ , namely  $\alpha_{Rn}(r) = \alpha_{Ln}(r) \sim \exp[-2A(r)]$ , which are not normalizable.

Now, in order to compare our results with a previous one [31], we address the issue of possible localization of massless fermions on the brane by setting the general coupling

$$F(\phi, \chi) = \omega_1 \phi + \omega_2 \chi + \omega_3 \phi \chi. \quad (44)$$

to the fermions, where the  $\omega_i$ 's are constant parameters to be determined such that  $\alpha_{R0}(r)$  or  $\alpha_{L0}(r)$  is normalizable. We have analyzed the scenarios of both degenerate and critical branes for  $\lambda = \mu$ .

It is important to remark that the factor  $e^{-A(r)}$  (see eq. (37)) diverges for  $r \rightarrow \pm\infty$  in the case of DBW, such that it does impose restrictive conditions on the normalization of the wave functions. On the other hand, in the case of CBW, the factor  $e^{-A(r)}$  diverges for  $r \rightarrow -\infty$  and is constant for  $r \rightarrow \infty$  (we have taken the upper signs in expressions (12)-(14)). Thus, the normalization of the wave function can only be found by means of a fine tuning on the  $\omega_i$ 's.

In the case of CBW with  $\lambda = \mu$  one can see from (12) and (13) that the coupling  $\omega_3 \phi \chi$  contributes to a constant for the behavior of  $-2\eta \int^r F(r') dr'$  for  $r \rightarrow \pm\infty$ . In fact, we have found that

$$\exp(-2\eta \int^r F(r') dr') \sim \begin{cases} \exp(-2\eta a \omega_2 r), & r \rightarrow +\infty \\ \exp(-2\eta a \omega_1 r), & r \rightarrow -\infty \end{cases}, \quad (45)$$

whilst  $e^{-A(r)} \sim e^{4a^3 \mu |r|/9}$  for  $r \rightarrow -\infty$ . Then by choosing  $\eta > 0$ ,  $\omega_2 > 0$  and  $\omega_1 < -2a^2 \mu / 9\eta$  one has localized left-handed massless fermion. On Figure 4 we show the profile for the  $r$ -dependent part of the coefficient of the spinor for  $\omega_1 = -1/3, -1, -3$  and  $\omega_3 = 0$ ,  $\eta = \omega_2 = a = \mu = 1$ . As a conclusion, the coupling of fermions with the field  $\phi(r)$  is relevant to provide localized massless fermions, but  $\omega_1$  should be close to  $-2a^2 \mu / 9\eta$  to insure a sharp localization in the core of the wall, since the peak of  $\alpha_{L0}(r)$  dislocates to the right of the core of the wall as  $|\omega_1|$  increases. Moreover, the dominant contribution for  $r \rightarrow +\infty$  comes from the coupling to the field  $\chi(r)$ , which describes the internal structure of the brane, and we have find that the coupling constant  $\omega_2$  must be positive in order to insure the normalization of the wave function associated to the massless fermion. We have also checked that the coupling  $\omega_3 \phi \chi$  contributes to dislocate the peak of  $\alpha_{L0}(r)$  from the core of the wall and that this coupling does not insure the fermion localization by itself, as has already been shown in [35]. Our results are in complete agreement with those found in reference [31].

In the case of CBW with  $\lambda = 4\mu$  one has the following asymptotic behaviors

$$\begin{aligned} \exp(-2\eta \int^r F(r') dr') &\sim \begin{cases} \exp(-4\eta a \omega_2 r), & r \rightarrow +\infty \\ \exp(-4\eta a \omega_1 r), & r \rightarrow -\infty \end{cases}, \\ e^{-A(r)} &\sim \begin{cases} \text{const.}, & r \rightarrow +\infty \\ \exp(-\frac{16a^3\mu r}{9}), & r \rightarrow -\infty \end{cases}. \end{aligned} \quad (46)$$

such that one can choose  $\eta > 0$ ,  $\omega_2 > 0$  and  $\omega_1 < -4a^2\mu/9\eta$  in order to have normalizable fermion zero modes trapped inside the wall.

We have also considered the localization of massless fermions on DBW with  $\lambda = 4\mu$  by taking the solutions (9) and (10). We have found that smooth normalizable solutions are obtained by setting  $\omega_2 = \omega_3 = 0$  and  $\omega_1 > 8a^2\mu/9\eta$ . We show in Figure 5 the behavior of  $\alpha_{L_0}(r)$  for two different values of  $c_0$ ,  $c_0 \ll 1/16a^2$  and  $c_0 \lesssim 1/16a^2$ . One can note that the behavior of  $\alpha_{L_0}(r)$  does not follow the behavior of the brane. Whilst Figure 3 shows the two-kink solutions, which indicates the brane splitting, Figure 5 shows that the left handed massless fermion is likely to be seen in the bulk between the two walls as  $c_0$  approaches the critical value. Although this seems to be a democracy in the sense that there is no preferable wall for the fermion to be trapped in, it also brings a paradox since the fermion lives in the bulk between the two walls preventing it to be measured. We have been looking for a solution to this apparent paradox. We have found that a coupling, reminiscent from supersymmetry, namely

$$F(\phi, \chi) = \omega_1 \frac{W_{\phi\phi} W_{\phi} + W_{\phi\chi} W_{\chi}}{W_{\phi}} = \omega_1 \frac{\phi''}{\phi'}, \quad (47)$$

with  $\omega_1 < -2a^2/9\eta$  can afford localized massless fermion with definite chirality whose  $r$ -dependent behavior follows the brane splitting by exhibiting a sharp localization in the cores of the walls as can be seen in Figure 6. In this scenario the massless fermion can be found in both walls simultaneously and the probability density for the fermion to be found between both walls diminishes as the walls are far apart from each other.

## 4 Conclusions

In this work we analyze the localization of massless fermions on degenerate and critical Bloch branes. The calculations are done directly on the original extra-dimensional spatial variable, that is, the physical coordinates, in contrast to some works that resort to the transformation of variables  $dz = e^{-A(r)} dr$ , that is, the conformal coordinates, in order to obtain effective time-independent Schrödinger equations, for the  $z$ -dependent parts of the spinors, with eigenvalue equals to the fermion mass. With such a procedure the effective Schrödinger equation allows us to see the emergence of a possible Kaluza-Klein (KK) tower of massive fermions. On the other hand, the appearance of possible non-localized KK modes is not evident from the effective Schrödinger equations in the physical coordinate  $r$ , equations (40)-(43). This is due to the fact that the mass contributes to the effective potentials, i.e., for each mass one has a different potential, such that one should look for scattering states, or even massive fermion localized states, by means of an effective Schrödinger equation with eigenvalue zero.

Notwithstanding, one does not have to resort to an effective Schrödinger equation in order to analyze the existence of normalizable localized states of massless fermions.

It is important to remark that the interaction term of fermions with the scalar field is crucial to the correct localization of fermions on the brane. The most natural coupling is the Yukawa one, namely,  $\phi\bar{\Psi}\Psi$ , used originally in [11] to explain the charge fractionization by a soliton background, and established in [1] to illustrate localization of fermions in a domain-wall. In fact, the Yukawa coupling is the simplest one, but such a choice comes naturally if one has in mind the potential  $\frac{\lambda}{2}(\phi^2 - 1)^2$ , whose BPS solution is the soliton background that traps the fermions. In this case, the Yukawa coupling entails a  $N = 1$  supersymmetry (SUSY) in the fermion-boson system, once the superpotential is  $W(\phi) = \sqrt{\lambda}\phi(\frac{\phi^2}{3} - 1)$  and  $d^2W/d\phi^2 = 2\sqrt{\lambda}\phi$ . In the context of branes in a warped space-time the issue of SUSY, that is supergravity, is much more complicate, moreover when it comes with two scalar fields, which is the case of Bloch branes in a model which supports a variety of soliton solutions. We have shown that the general coupling  $(\omega_1\phi + \omega_2\chi + \omega_3\phi\chi)\bar{\Psi}\Psi$  guarantees the localization of massless fermions, for a range of the coupling constants  $\omega_1$ ,  $\omega_2$  and  $\omega_3$ , and that the cross term  $\phi\chi$  does not provide fermion localized states by itself, in agreement with previously reported calculations [31], [35]. The general coupling also works even when one has two-kink solutions, but the  $r$ -dependent part of the wave-fuction is peaked just in the middle of the region between the walls. Such a behavior is not the desirable one, once it signalizes that the fermions would not be observed inside the branes. We have proposed another coupling between fermion and scalar fields which provides the correct localization of fermionic zero-modes inside the branes. The chosen coupling seems to come as a reminiscent of a SUSY model. This last issue has been analyzed in [36].

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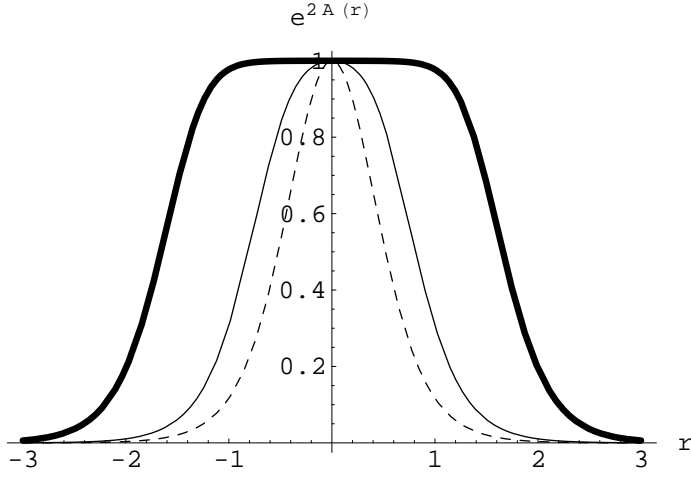


Figure 1: Warp factor in the case of DBW with  $\lambda = 4\mu$ :  $c_0 = 0$  (dashed line),  $c_0 = 1/17$  (thin solid line),  $c_0 = 1/16.001$  (thick solid line) with  $a = \mu = 1$ .

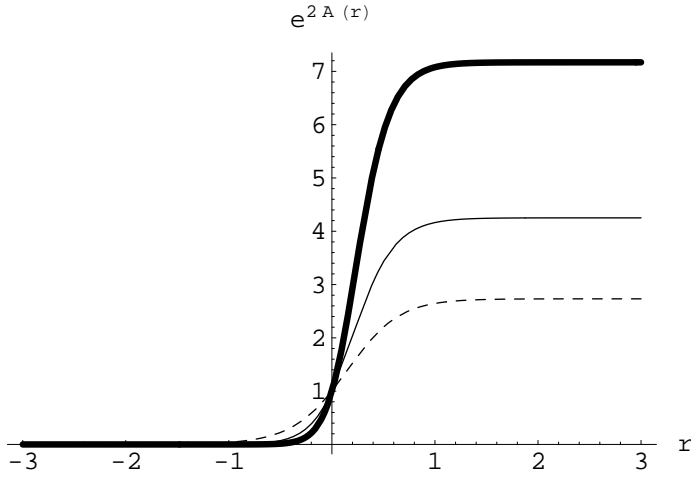


Figure 2: Warp factor in the case CBW with  $\lambda = 4\mu$  for  $\mu = 1$  and  $a = 1$  (dashed line),  $a = 1.2$  (thin solid line),  $a = 1.4$  (thick solid line).

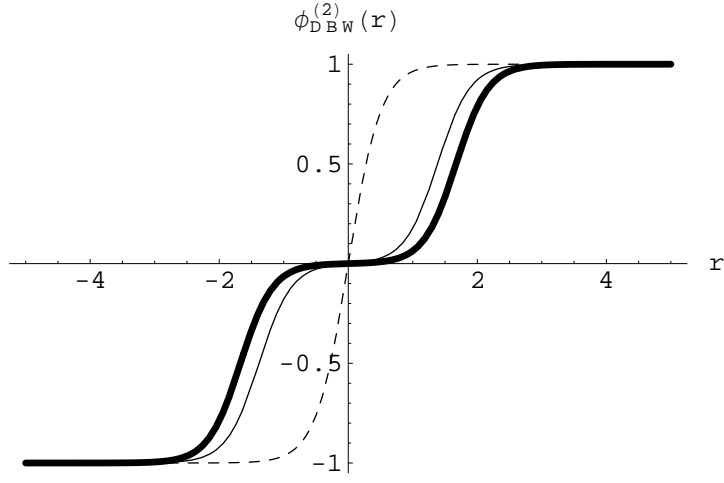


Figure 3: Two-kink solution (DBW) with  $\lambda = 4\mu$ :  $c_0 = 0$  (dashed line),  $c_0 = 1/16.001$  (thin solid line),  $c_0 = 1/16.0001$  (thick solid line) with  $a = \mu = 1$ .

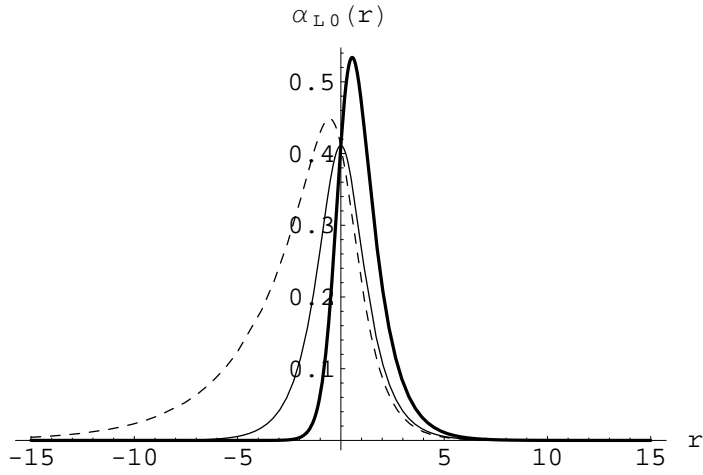


Figure 4:  $\alpha_{L0}(r)$  in the case of CBW with  $\lambda = 4\mu$  and  $\omega_2 = 1$ ,  $\omega_3 = 0$ ,  $\omega_1 = -1/3$  (dashed line),  $\omega_1 = -1$  (thin solid line) and  $\omega_1 = -3$  (thick solid line);  $a = \mu = 1$ .

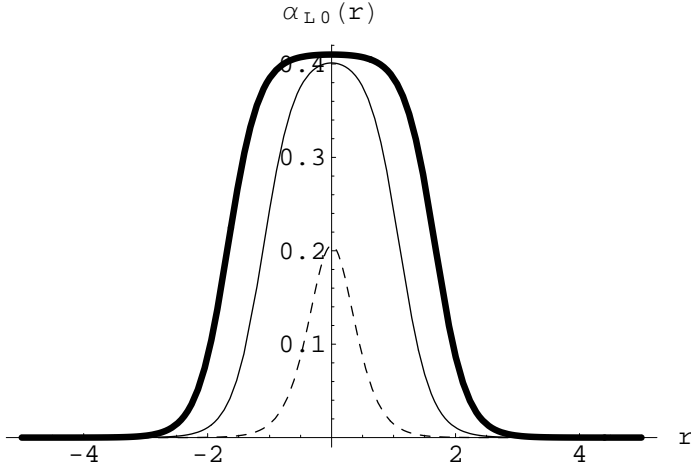


Figure 5:  $\alpha_{L0}(r)$  in the case of DBW ( $\lambda = 4\mu$ ) and  $F = \omega_1\phi$  for  $\omega_1 = 4$ ,  $c_0 = 0$  (dashed line),  $c_0 = 1/16.01$  (thin solid line) and  $c_0 = 1/16.0001$  (thick solid line);  $a = \mu = \eta = 1$ .

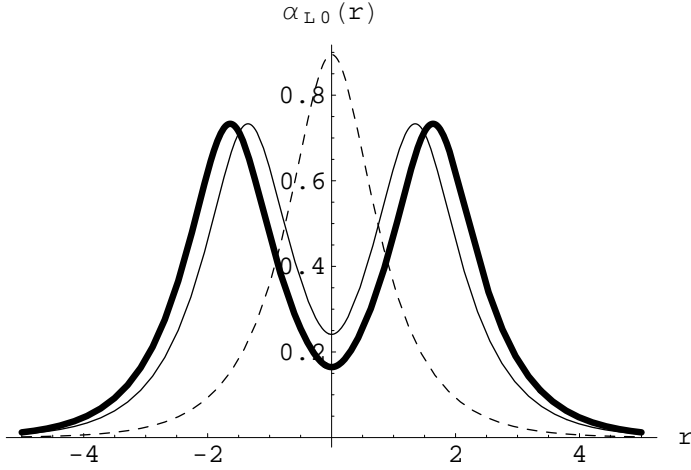


Figure 6:  $\alpha_{L0}(r)$  in the case of DBW ( $\lambda = 4\mu$ ) and  $F = \omega_1\phi''/\phi'$  for  $\omega_1 = -1/3$ ,  $c_0 = 0$  (dashed line),  $c_0 = 1/16.001$  (thin solid line) and  $c_0 = 1/16.0001$  (thick solid line);  $a = \mu = \eta = 1$ .